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Some Convergence Results for Partial Maps

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Abstract. In this paper we analyze some aspects of a new notion of convergence for nets of partial maps, introduced in [8]. In particular, we show that the introduced bornological convergence reduces to a natural uniform convergence relative to the bornology when the partial maps have a common domain. We then provide a new notion of upper convergence, which looks much more manageable than the original one. We show that the two notions, though different in general cases, do agree for *sequences* of strongly uniformly continuous (relative to the bornology) partial maps. More generally, coincidence for nets is shown in case the target space of the maps is totally bounded. This last result is interesting in view of possible applications, since partial maps are usually utility functions, thus when dealing with general models, monotone transformations valued in $[0, 1]$ give rise to the same utility functions.

1. Introduction

The notion of partial map probably goes back to Kuratowski [15], but only in the last decades the study of topologies and convergences on partial maps started to be developed ([2], [7] [9] and [8]), mainly from the point of view of applications ([1], [7] [2], [6], and [16]). In mathematical economics a partial map with codomain $\mathbb{R}$ represents a utility function, and in [2] K. Back introduced the so called generalized compact-open topology on the space of these partial maps, in order to define *similarity* among economic agents. His topology topologizes a generalized continuous convergence and it is used for applications in dynamic programming models ([16]). In [7] it was proved that under some conditions this topology coincides with a topology introduced in [6] in the setting of differential equations. Holá ([11] and [12]) characterized its main topological properties. We want to explicitly observe that these topologies on partial maps are different in spirit from the usual variational convergences, like Γ-convergences, the Mosco and the Attouch-Wets convergences and so on. The key difference is in the behavior of the domains. When dealing with variational convergences one usually considers functions not necessarily defined on the whole space, by extending them outside the effective domain with an appropriate value (∞ in the case of cost functions, $-\infty$ for utility functions), and then by defining a topology/convergence via some set convergence of epigraphs (cost functions) or hypographs (utility functions). This however in general implies only weak convergence modes for the domains: for instance the sequence $f_n(x) = nx^2$, everywhere defined, converges in any natural sense to the indicator function of the origin, having thus a singleton as effective domain. This is not suitable in an economic context: since the domain of the utility function is the set on which an agent

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is able to express his preferences, it is clear that we cannot consider as similar agents having quite different preference sets. And in the example above the domain of the limit map is $\{0\}$, while the approximating maps have as domain the whole space. For this reason topologies on partial maps usually require specific conditions on the topology inherited by the domains of the functions.
Starting from the first seminal result by Back, as mentioned before several papers were considering analogous topologies/convergences, essentially when the domain space is locally compact. An attempt to find a new general definition of topology on the family of all partial maps appears in [9], in a proximity setting. Even if this definition is more general than the previous one, no characterization appears on the behaviour of the values of the functions, so that it turns out to be significant from a practical point of view only when it coincides with the generalized compact-open topology.

In the very recent paper [8], Beer *et al.* proposed a new approach to define convergences on partial maps, in the metric setting, by providing a very general definition of convergence on the family of partial maps with closed domain, through the notion of bornology. In this paper we continue the study of this new convergence, and we propose a new, simpler definition of (upper) convergence, that is coarser than the general one. However we show that in a broad and natural subclass of partial maps convergence of sequences is the same in the two settings. Furthermore we show that the two upper convergences do coincide when the target space is totally bounded. This result is interesting for applications, since monotone transformations of a utility function provide other utility functions. Thus, unless some specific feature of the function (like convexity, for instance) must be preserved, one can work with $[0, 1]$-valued functions.

We conclude by observing that a partial map is characterized by its *domain*, and by the *values* it assumes on the target space. Since we shall always assume that the maps have closed domains, our definitions of convergences on partial maps subsume and include definitions of convergences on the closed nonempty subsets of the metric space X. This can be easily seen by considering as a target space Y just a one-point space, or alternatively for arbitrary Y by considering the subset of the partial map space consisting of all maps having as a codomain a fixed $y \in Y$.

2. Notations and Preliminaries

Throughout the paper $X = (X, d)$ and $Y = (Y, \rho)$ will denote metric spaces. We write $CL(X)$ for the collection of the closed nonempty subsets of X; $K(X)$ is the collection of the compact nonempty subsets of X. If $x_0 \in X$ and $\epsilon > 0$, $B[x_0, \epsilon]$ is the open ϵ-ball with center x_0 and radius ϵ. If A is a nonempty subset of X, we write $d(x_0, A)$ for the distance from x_0 to A. We denote by A^ϵ the *ϵ-enlargement* of the set A:

$$A^\epsilon = \{x : d(x, A) < \epsilon\} = \bigcup_{x \in A} B[x, \epsilon].$$

We now introduce the notion of bornology (see [10] and [13]).

Definition 2.1. *A* bornology $\mathcal{B}$ *on a metric space* (X, d) *is a family of subsets of* X*, covering* X*, closed under taking finite unions, and hereditary.*

The smallest bornology on X is the family of the finite subsets of X, $\mathcal{F}$, and the largest is the family of all non empty subsets of X, $\mathcal{P}_0(X)$. Other important bornologies are: the family $\mathcal{B}_d$ of the nonempty d-bounded subsets, the family $\mathcal{B}_{tb}$ of the nonempty d-totally bounded subsets and the family $\mathcal{K}$ of nonempty subsets of X with compact closure.
Bornological convergence as defined in [17] is split into upper and lower bornological convergence. Here are the basic definitions. Let Γ be a set directed by $\geq$ (see [14] for basics on convergence of nets).

Definition 2.2. *A net* $\langle D_\gamma \rangle_{\gamma \in \Gamma}$ *in* $\mathcal{P}_0(X)$ *is* $\mathcal{B}^-$*-convergent to* $D \in \mathcal{P}_0(X)$ *if*

$$\forall \epsilon > 0, \forall B \in \mathcal{B}, \exists N \in \Gamma : \forall \gamma > N \quad D \cap B \subset D_\gamma^\epsilon.$$

Definition 2.3. *A net $\langle D_\gamma \rangle_{\gamma\in\Gamma}$ in $\mathcal{P}_0(X)$ is $\mathcal{B}^+$-convergent to $D \in \mathcal{P}_0(X)$ if*

$$\forall \epsilon > 0, \forall B \in \mathcal{B}, \exists N \in \Gamma : \forall \gamma > N \quad D_\gamma \cap B \subset D^\epsilon.$$

$\mathcal{B}^-$-convergence is called *lower bornological convergence*, $\mathcal{B}^+$-convergence is called *upper bornological convergence* and the join of the two is called *(two-sided) bornological convergence*. The relative notation will be: $D \in \mathcal{B} - \lim D_\gamma$.
For more on bornological convergence of sets see [4] and [17].

We now provide the fundamental definition of partial map between metric spaces.

Definition 2.4. *A* partial map *between the metric spaces X and Y is a pair (D, u), where $D \in CL(X)$, and $u : D \to Y$ is a map.*

In case we are considering maps with a fixed, common domain, we just denote it without explicitly mentioning the domain. Moreover, we shall write $\mathcal{P}[X, Y]$ for the set of all partial maps from X to Y. By $C[X, Y]$ we denote the family of all continuous partial maps. Throughout the paper, we shall extensively consider important subsets of $C[X, Y]$. To introduce them we need the following definitions (see [8]), which parallel those given in [5].

Definition 2.5. *Let $\mathcal{B}$ be a bornology on X and $(D, u) \in \mathcal{P}[X, Y]$. We say that (D, u) is* uniformly continuous relative to the bornology *$\mathcal{B}$ if for every $B \in \mathcal{B}$ the map $u : D \cap B \to Y$ is uniformly continuous. We say that (D, u) is* strongly uniformly continuous relative to the bornology *$\mathcal{B}$ if for every $B \in \mathcal{B}$ and for each $\epsilon > 0$ there is $\delta > 0$ such such that if $d(x, y) < \delta$ and $x, y \in B^\delta \cap D$, then $\rho(u(x), u(y)) < \epsilon$.*

We shall denote by $C_u(\mathcal{B})[X, Y]$ and $C_{su}(\mathcal{B})[X, Y]$ the set of all partial functions which are uniformly continuous and strongly uniformly continuous relative to $\mathcal{B}$, respectively.

In [8], Beer *et al*. introduced a new fundamental family of convergences on $\mathcal{P}[X, Y]$. Let Γ be a set directed by $\geq$.

Definition 2.6. *[8] Let (X, d) and (Y, ρ) be metric spaces. Let $\mathcal{B}$ be a bornology on X. A net $\langle (D_\gamma, u_\gamma) \rangle_{\gamma\in\Gamma}$ in $\mathcal{P}[X, Y]$ is said to be $\mathcal{P}^-(\mathcal{B})$*-convergent *to (D, u) if*

$$\forall B \in \mathcal{B}, \forall \epsilon > 0 \ \exists N \in \Gamma : \forall \gamma \geq N, \forall B_1 \subset B, \ u(D \cap B_1) \subset [u_\gamma(D_\gamma \cap B_1^\epsilon)]^\epsilon.$$

Definition 2.7. *[8] Let (X, d) and (Y, ρ) be metric spaces. Let $\mathcal{B}$ be a bornology on X. A net $\langle (D_\gamma, u_\gamma) \rangle_{\gamma\in\Gamma}$ in $\mathcal{P}[X, Y]$ is said to be $\mathcal{P}^+(\mathcal{B})$*-convergent *to (D, u) if*

$$\forall B \in \mathcal{B}, \forall \epsilon > 0 \ \exists N \in \Gamma : \forall \gamma \geq N, \forall B_1 \subset B, \ u_\gamma(D_\gamma \cap B_1) \subset [u(D \cap B_1^\epsilon)]^\epsilon.$$

When $\langle (D_\gamma, u_\gamma) \rangle$ is $\mathcal{P}^-(\mathcal{B})$-convergent to (D, u), we shall write $(D, u) \in \mathcal{P}^-(\mathcal{B}) - \lim(D_\gamma, u_\gamma)$. When $\langle (D_\gamma, u_\gamma) \rangle$ is $\mathcal{P}^+(\mathcal{B})$-convergent to (D, u), we shall write $(D, u) \in \mathcal{P}^+(\mathcal{B}) - \lim(D_\gamma, u_\gamma)$. The join of these convergences will be denoted by $(D, u) \in \mathcal{P}(\mathcal{B}) - \lim(D_\gamma, u_\gamma)$.

In [8] it is proved that $\mathcal{P}^-(\mathcal{B})$-convergence implies lower bornological convergence of domains, while $\mathcal{P}^+(\mathcal{B})$-convergence implies upper bornological convergence of domains.

The above definition can be reformulated in some equivalent ways.

Remark 2.8. [8] On $\mathcal{P}[X, Y]$

- the condition

$$\forall B \in \mathcal{B}, \forall \epsilon > 0, \exists N \in \Gamma : \forall \gamma \geq N, \forall B_1 \subset B, u(D \cap B_1) \subset [u_\gamma(D_\gamma \cap B_1^\epsilon)]^\epsilon$$

  in Definition 2.6 is equivalent to the following condition:

$$\forall B \in \mathcal{B}, \forall \epsilon > 0, \ \exists N \in \Gamma : \forall \gamma \geq N, \ \sup_{z\in D\cap B} \inf_{x\in B[z,\epsilon]\cap D_\gamma} \rho(u_\gamma(x), u(z)) < \epsilon.$$

- Dually, the condition

$$\forall B \in \mathcal{B}, \forall \epsilon > 0,\ \exists N \in \Gamma : \forall \gamma \geq N, \forall B_1 \subset B,\ u(D_\gamma \cap B_1) \subset [u(D_\gamma \cap B_1^\epsilon)]^\epsilon$$

  in Definition 2.7 is equivalent to the following condition:

$$\forall B \in \mathcal{B}, \forall \epsilon > 0,\ \exists N \in \Gamma : \forall \gamma \geq N,\ \sup_{z \in D_\gamma \cap B} \inf_{x \in B[z,\epsilon] \cap D} \rho(u(x), u_\gamma(z)) < \epsilon.$$

3. Convergence for Particular Classes of Partial Maps

We start this section with a result, that describes the introduced convergence in the case when the partial maps have a common domain. Next, we introduce a new definition of upper convergence. As already remarked, the original definition looks a bit complicated, since it requires the fulfillment of a condition in a uniform way on the subsets of a given set. Thus we consider the coarser case when the condition is required only on the set (and not uniformly on the subsets). We see by examples that this gives rise to a different convergence notion. However we can prove, and these are our main results, that the notions agree when sequences are considered instead of nets, and coincide for nets either for particular bornologies or for particular target spaces Y.

In order to establish our first result, we need to recall the following.

Proposition 3.1. *[8] Let $(D, u) \in \mathcal{P}[X, Y]$ be a partial map strongly uniformly continuous relative to the bornology $\mathcal{B}$. Let Γ be a set directed by $\geq$. A net $\langle (D_\gamma, u_\gamma) \rangle_{\gamma \in \Gamma}$ in $\mathcal{P}[X, Y]$ is $\mathcal{P}^+(\mathcal{B})$-convergent to (D, u) if and only if*

$$\forall B \in \mathcal{B} \forall \epsilon > 0, \exists \delta > 0, \exists N \in \Gamma : \forall \gamma \geq N,\ \sup_{z \in D_\gamma \cap B} \sup_{x \in B[z,\delta] \cap D} \rho(u(x), u_\gamma(z)) < \epsilon.$$

We are ready to prove the first announced result.

Theorem 3.2. *Let Γ be a set directed by $\geq$. Suppose $\langle (D, u_\gamma) \rangle_{\gamma \in \Gamma}$ is a net of partial maps from $\mathcal{P}[X, Y]$ (with a common domain). Then*

(1) if (D, u_γ) uniformly converges to (D, u) on the bornology $\mathcal{B}$, then it $\mathcal{P}(\mathcal{B})$-converges;

(2) the converse is true provided (D, u) is strongly uniformly continuous relative to $\mathcal{B}$.

Proof In the proof we shall not refer to the domain D, since it is fixed. (1) easily follows from the fact that for all $B \in \mathcal{B}$ and all $\epsilon > 0$ it holds:

$$\sup_{z \in D \cap B} \rho(u_\gamma(z), u(z)) \geq \max\{ \sup_{z \in D \cap B} \inf_{x \in B[z,\epsilon] \cap D} \rho(u_\gamma(z), u(x)), \sup_{z \in D \cap B} \inf_{x \in B[z,\epsilon] \cap D} \rho(u_\gamma(x), u(z)) \}.$$

Now let us see (2). It is obvious that

$$\sup_{z \in D \cap B} \rho(u_\gamma(z), u(z)) \leq \sup_{z \in D \cap B} \sup_{x \in B[z,\epsilon] \cap D} \rho(u_\gamma(z), u(x)).$$

Since u is strongly uniformly continuous relative to $\mathcal{B}$, from $\mathcal{P}(\mathcal{B})$-convergence and by Proposition 3.1 it holds that $\sup_{z \in D \cap B} \sup_{x \in B[z,\epsilon]} \rho(u_\gamma(z), u(x)) < \epsilon$ eventually for all $\epsilon > 0$, and thus the proof is concluded. ■

We now consider an example clarifying the role of the assumptions in the above result. In particular we show that in (2) the assumption of strongly uniform continuity for (D, u) cannot be weakend to continuity.

Example 3.3. Let X be a separable real Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and orthonormal base $\{e_n : n \in \mathbb{N}\}$. Let $\mathcal{B}$ be the bornology of all subsets B of X of the form $B = H \cup K$, where H is a subset of the closed unit ball and K is a finite set. Finally, let

$$u(x) = \sum_{k=1}^{\infty} \langle x, e_k \rangle^{2k}, \quad u_n(x) = u((1 + \frac{1}{n})x).$$

Then the following facts hold:

- $u_n(e_n) - u(e_n) = (1 + \frac{1}{n})^{2n} - 1$ which implies that u_n does not converge uniformly to u on the unit ball;
- $u_n(x) - u((1 + \frac{1}{n})x) = 0$, which implies that u_n is $\mathcal{P}(\mathcal{B})$-convergent to u.

It is time now to propose the new definition of (upper) convergence on partial maps. This definition is inspired by the approach given in [9], when an attempt was made to extend the generalized compact open topology to a non locally compact setting. However since a compact open topology should be equivalent in general to uniform convergence, the previous result shows that the right generalization of Back's approach is $\mathcal{P}(\mathcal{B})$-convergence. But we want to consider also the following definition since it is simpler than $\mathcal{P}(\mathcal{B})$-convergence, and for some important cases the two convergences agree.

Definition 3.4. *Let $(X, d), (Y, \rho)$ be metric spaces, and let $\mathcal{B}$ be a bornology on X. Let Γ be a set directed by $\geq$ and let $\langle (D_\gamma, u_\gamma) \rangle_{\gamma \in \Gamma}$ be a net in $\mathcal{P}[X, Y]$. We say that the net is $\mathcal{M}^+(\mathcal{B})$-*convergent *to (D, u), and write $(D, u) \in \mathcal{M}^+(\mathcal{B})$-$\lim(D_\gamma, u_\gamma)$, if*

$$\forall B \in \mathcal{B}, \forall \epsilon > 0, \exists \delta > 0, \exists N \in \Gamma : \forall \gamma \geq N, \ u_\gamma(D_\gamma \cap B) \subset [u(D \cap B^\epsilon)]^\epsilon.$$

As a first remark, we want to observe that the above condition implies also upper bornological convergence of the domains as it is shown in the following proposition.

Proposition 3.5. *Let $\langle (D_\gamma, u_\gamma) \rangle_{\gamma \in \Gamma}$ be a net in $\mathcal{P}[X, Y]$. Suppose $\langle (D_\gamma, u_\gamma) \rangle_{\gamma \in \Gamma}$ is $\mathcal{M}(\mathcal{B})^+$-convergent to (D, u). Then for every $\epsilon > 0$ eventually $D_\gamma \cap B \subset D^\epsilon$.*

Proof Suppose not. Then for every γ_0 there are $\gamma \geq \gamma_0$ and $x_\gamma \in B \cap D_\gamma$ such that $d(x_\gamma, D) > \epsilon$. Call B_1 the set of these elements x_γ, and observe that $B_1 \in \mathcal{B}$. Since $\langle (D_\gamma, u_\gamma) \rangle_{\gamma \in \Gamma}$ is $\mathcal{M}^+(\mathcal{B})$-convergent to (D, u), then eventually it must be

$$u_\gamma(D_\gamma \cap B_1) \subset [u(D \cap B_1^\epsilon)]^\epsilon.$$

But this is impossible, since $D \cap B_1^\epsilon = \emptyset$. ■

It is clear that $\mathcal{P}^+(\mathcal{B})$-convergence implies $\mathcal{M}^+(\mathcal{B})$-convergence. Moreover, it follows from Theorem 4.4. in [8], that in $C[X, Y]$ $\mathcal{M}^+(\mathcal{K})$-convergence coincides with $\mathcal{P}^+(\mathcal{K})$-convergence. But what happens outside the compact case? In order to give an insight to the question, let us consider the following examples.

Example 3.6. Let $X = \cup_{k=1}^{\infty} X_k$, where $X_k = [(k-1)\pi, (k-1)\pi + \frac{\pi}{2k})$. In what follows the functions are defined and continuous everywhere, so that we do not refer to their domains. Let $u, u_n, n = 1, 2, \ldots$ be the following functions: $u(x) = \min\{\tan kx, k\}$ if $x \in X_k$, and

$$u_n(x) := \begin{cases} u(x) & \text{if} \quad x \notin X_n \\ \tan nx & \text{if} \quad x \in X_n. \end{cases}$$

We claim that u_n is $\mathcal{M}^+(\mathcal{B})$ convergent to u, but not $\mathcal{P}^+(\mathcal{B})$ convergent with $\mathcal{B} = \mathcal{P}_0(X)$. Let us show $\mathcal{M}^+(\mathcal{B})$-convergence. Fix $B \in \mathcal{B}$ and $\epsilon > 0$. There are two cases: 1) B is contained in a finite number of

$X_{k_1},...,X_{k_j}$. Then for all $x \in B$ it is $u(x) = u_n(x)$ for all $n > \max\{k_1,\ldots,k_j\}$ and the conclusion is obvious. 2) Suppose there is subsequence X_{k_n} such that B has points in every X_{k_n}. Then B^ϵ contains X_{k_n} for all large n, so that $u(B) = [0,+\infty)$ and the conclusion easily follows. Now we prove that the sequence does not $\mathcal{P}^+(\mathcal{B})$-converge. For, take $B = X$ and $\epsilon = \frac{\pi}{4}$. If the sequence $\mathcal{P}^+(\mathcal{B})$ converged, there would be N such that for all $n > N$ and for all $L \subset X$ it is $u_n(L) \subset [u(L^\epsilon)]^\epsilon$. But if we consider $L = X_n$ we have that $u_n(L) = [0,+\infty)$, while $u(L) = [0,n]$, and this concludes the example.

The next example shows that the two convergences need not to coincide when considering continuous functions defined on a complete metric space.

Example 3.7. Let H be a separable Hilbert space, let $\{e_k, k \in \mathbb{N}\}$ be an orthonormal basis, let $X_k = B[e_k; \frac{1}{2\sqrt{k}}]$, let $X = \cup_{k=1}^\infty X_k$, metrized by the norm in X and let $\mathcal{B} = \mathcal{P}_0(X)$. Let

$$u(x) = \sum_{j=1}^{k} k(x,e_j)^{2j} \quad x \in X_k,\ k \geq 1$$

and

$$u_n(x) := \begin{cases} u(x) & \text{if} \quad x \notin X_n \\ u(x) + \displaystyle\sum_{j=1}^{\infty} j(\sqrt{n}(x-e_n),e_j)^{2j} & \text{if} \quad x \in X_n. \end{cases}$$

The fact that the sequence u_n is $\mathcal{M}^+(\mathcal{B})$-convergent to u but not $\mathcal{P}^+(\mathcal{B})$-convergent follows the line of proof of the previous example; in particular if $B \subset X$ hits every member of some subsequence X_{n_k}, then, given a fixed $\epsilon > 0$, there is k_0 so that B^ϵ contains B_{n_k} for all $k > k_0$. It follows that

$$u(B^\epsilon) \supset \cup_{k\geq k_0} u(X_{n_k}) \supset \cup_{k\geq k_0}[1,(1+\frac{1}{2\sqrt{n_k}})^{2n_k}] = [1,+\infty)$$

which implies $\mathcal{M}^+(\mathcal{B})$-convergence, since $u_n \geq 1$. To show that $\mathcal{P}^+(\mathcal{B})$-convergence does not occur as in the previous example one can see that $u_n(X_n)$ is upper unbounded while $u(X_n)$ is not.

Now a natural question arises: are there meaningful cases for which $\mathcal{M}^+(\mathcal{B})$ and $\mathcal{P}^+(\mathcal{B})$ do coincide? The rest of the paper is dedicated to provide some results concerning this issue.

Observe that in Example 3.7 the involved maps are *not* strongly uniformly continuous relative to the bornology. Our first basic result shows that, at least as far as convergence of *sequences* is concerned, strong uniform continuity of the maps suffices.

Theorem 3.8. *Let X be a metric space, let $\mathcal{B}$ be a bornology. Then on $C_{su}(\mathcal{B})[X,Y]$, $\mathcal{M}^+(\mathcal{B})$-convergence of sequences coincides with $\mathcal{P}^+(\mathcal{B})$-convergence.*

Proof The proof goes by contradiction and is divided into several steps. So, let us assume there is a sequence (D_n,u_n) converging to (D,u) for $\mathcal{M}^+(\mathcal{B})$ and not for $\mathcal{P}^+(\mathcal{B})$. Then $\mathcal{M}^+(\mathcal{B})$ convergence implies that for every $\hat{B} \in \mathcal{B}$, $\sigma > 0$ and $x_n \in \hat{B} \cap D_n$, eventually

$$u_n(x_n) \in [u(D \cap \hat{B}^\epsilon)]^\epsilon. \tag{1}$$

Moreover, since (D_n,u_n) does not $\mathcal{P}^+(\mathcal{B})$-converge to (D,u), there are $B \in \mathcal{B}$ and $\epsilon > 0$ so that, by possibly passing to subsequences, for every n there is $L_n \subset B$ such that $u_n(L_n \cap D_n) \cap [[u(D \cap [L_n]^\epsilon)]^\epsilon]^c \neq \emptyset$; thus there is $x_n \in L_n \cap D_n$ such that

$$u_n(x_n) \notin [u(D \cap [L_n]^\epsilon)]^\epsilon.$$

We now prove that this last relation leads to a contradiction. We need to distinguish three cases:

1. (x_n) has a Cauchy subsequence

2. $u_n(x_n)$ has a Cauchy subsequence
3. (x_n) and $u_n(x_n)$ have a common discrete subsequence.

Case 1.
There is (x_n) as above such that it has a Cauchy subsequence. W.l.o.g., as usual, we can suppose (x_n) itself to be Cauchy. Let us fix $\sigma > 0$ such that $2\sigma < \epsilon$. There is N such that, for $n, m \geq N$, $d(x_n, x_m) < \sigma$. Set $\hat{B} = \cup_{k \geq N}\{x_k\} \in \mathcal{B}$. Then eventually

$$u_n(x_n) \in [u(\hat{B}^\sigma \cap D)]^\sigma \subset [u([L_n]^\epsilon \cap D)]^\epsilon$$

a contradiction.

Case 2.
Now suppose that every x_n as above is such that $u_n(x_n)$ has a Cauchy subsequence. We suppose, w.l.o.g., that there is N such that for all $n, m \geq N$, $\rho(u_n(x_n), u_m(x_m)) < \sigma$, where $0 < 2\sigma < \epsilon$. Denote by $\hat{B} \in \mathcal{B}$ the set $\hat{B} = \{x_k : k \geq N\}$ and, using (1), find $m, n \geq N$ so that

$$u_m(x_m) \in [u([x_n]^\sigma \cap D)]^\sigma.$$

It follows that

$$u_n(x_n) \in [u_m(x_m)]^\sigma \subset [[u([x_n]^\sigma \cap D)]^\sigma]^\sigma \subset [u([x_n]^\epsilon \cap D)]^\epsilon,$$

which contradicts (2).

Case 3.
The only remaining case to analyze is when there is (x_n) as above such that both (x_n) and $(u_n(x_n))$ are λ discrete for some $\lambda > 0$. Since $u \in C_{su}(\mathcal{B})[X, Y]$, we can find positive σ, τ, with $2(\sigma + \tau) < \lambda$, such that $d(x, y) < \sigma$, $x \in B$, imply $\rho(u(x), u(y)) < \tau$; moreover, since $\sigma < \lambda$, the sequence $\{x_n\}$ is σ-discrete. Set $\hat{B} = \{x_k : k \geq 1\} \in \mathcal{B}$ and use (1) for each $n \geq N$ to find k_n such that

$$u_n(x_n) \in [u([x_{k_n}]^\sigma) \cap D]^\sigma.$$

Note that (k_n) is a injective sequence, since if $k = k_m = k_n$ for some $n \neq m$, then

$$\rho(u_m(x_m), u_n(x_n)) \leq \operatorname{diam} u([x_k]^\sigma \cap D)^\sigma < 2(\tau + \sigma) < \lambda,$$

which would be a contradiction. Assume there is $N_0 \geq N$ such that $n \neq k_n$ whenever $n \geq N_0$, and inductively define the increasing sequence n_i of indices as follows: let $n_1 = N_0$. Given n_i for some $i \geq 1$, let

$$n_{i+1} = \min\{j > k_{n_i} : k_j \notin \{n_1, \dots, n_i\}\}.$$

Define $\hat{B}_0 = \{x_{n_i} : i \geq 1\} \in \mathcal{B}$ and use (1) to find i large enough that $u_{n_i}(x_{n_i}) \in [u([\hat{B}_0]^\sigma \cap D)]^\sigma$. However this is impossible, since otherwise, $u_{n_i}(x_{n_i}) \in [u([x_{n_j}]^\sigma \cap D)]^\sigma \cap [u([x_{k_{n_i}}]^\sigma \cap D)]^\sigma$ for some $n_j \neq k_{n_i}$, which contradicts σ- discreteness of the x_n's. It follows that $n = k_n$ frequently, so

$$u_n(x_n) \in [u([x_n]^\sigma \cap D]^\sigma \subset [u([L_n]^\sigma \cap D]^\sigma$$

for some $n \geq N_0$, which contradicts (2). This ends the proof. ■

The above result in particular holds for *any* bornology and *any* metric spaces. This means that it holds also when the hyperspace topology is not first countable. Thus coincidence of convergence of sequences is not equivalent to the coincidence of convergence of nets.

Our next result deals with coincidence for convergence of nets, for particular target spaces.

Theorem 3.9. *Let X be a metric space, $\mathcal{B}$ a bornology on X and suppose Y is totally bounded. Then on $\mathcal{P}[X, Y]$, $\mathcal{M}^+(\mathcal{B})$-convergence coincides with $\mathcal{P}^+(\mathcal{B})$-convergence.*

Proof Assume that there is a net (D_γ, u_γ) that is $\mathcal{M}^+(\mathcal{B})$-convergent to (D, u) but not $\mathcal{P}^+(\mathcal{B})$-convergent. Then we have that there are $\epsilon > 0$, $B \in \mathcal{B}$ and Γ_0 cofinal to Γ such that

$$\forall \gamma \in \Gamma_0 \exists L_\gamma \subset B, \exists x_\gamma \in L_\gamma \cap D_\gamma : u_\gamma(x_\gamma) \notin [u([L_\gamma]^\epsilon \cap D)]^\epsilon. \tag{2}$$

Let $\sigma > 0$ be such that $2\sigma < \epsilon$. Since Y is totally bounded, there are $y_1, \ldots, y_k \in Y$ such that $\cup_{i=1}^k B[y_i, \sigma] = Y$. Fix $i_0 \in \{1, 2, \ldots, k\}$ and a subset $\Gamma' \subset \Gamma$ cofinal to Γ_0 such that

$$u_\gamma(x_\gamma) \in B[y_{i_0}, \sigma] \quad \forall \gamma \in \Gamma'. \tag{3}$$

Claim

$$\forall \alpha \in \Gamma' \; \exists \beta \in \Gamma', \beta > \alpha \; : \; u([x_\beta]^\sigma \cap D) \cap B[y_{i_0}, \sigma] \neq \emptyset. \tag{4}$$

Now take a β from the Claim. Then (4) implies that

$$u_\beta(x_\beta) \in B[y_{i_0}, \sigma] \quad \wedge \quad u([x_\beta]^\sigma) \cap B[(y_{i_0}, 2\sigma] \neq \emptyset.$$

It follows that

$$u_\beta(x_\beta) \in [u([x_\beta]^\sigma) \cap D]^{2\sigma} \subset [u([x_\beta]^\epsilon) \cap D]^\epsilon,$$

contradicting Equation (2).

Thus in order to finish the proof we need to prove the Claim. We do it by contradiction. So, suppose there exists $\alpha \in \Gamma'$ such that, for all $\beta \in \Gamma'$, $\beta > \alpha$, it is

$$u([x_\beta]^\sigma \cap D) \cap B[y_{i_0}, \sigma] = \emptyset.$$

Set $\hat{B} = \cup_{\beta > \alpha}\{x_\beta\}$. Then there exists $\gamma_0 > \alpha$ such that for all $\gamma > \gamma_0$

$$u_\gamma(\hat{B} \cap D_\gamma) \subset [u([\hat{B}]^\sigma \cap D)]^\sigma.$$

Fix $\gamma > \gamma_0$. Then there is x_β, with $\beta > \alpha$, such that

$$u_\gamma(x_\gamma) \in B[y_{i_0}, \sigma] \cap [u([x_\beta]^\sigma \cap D)]^\sigma,$$

and this ends the proof. ∎

The last Theorem has particular relevance in the case when we consider real valued functions representing some preference systems of agents. In such a case since monotone transformations of utility function provide other utility functions for the same preference system, one can consider for instance as a target space Y the space $Y = [0, 1]$, and the above theorem applies.

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